Calculus II - Day 11

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Goals for today

- Review the Fundamental Theorem of Calculus
- Discuss the definite integral as a limit of Riemann sums
- Compute the total change given rate of change and apply to word problems

Derivatives: "rate of change"

$$\frac{d}{dt}f(t) = f'(t)$$

"Given a quantity that changes, how fast does it change?"

$\Rightarrow$ slope of tangent line

Integrals: "total change"

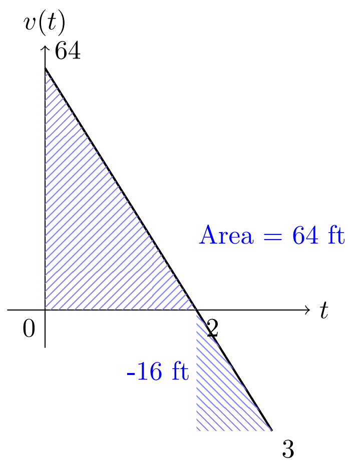
$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

"Given a rate of change, how much total change occurs?"

$\Rightarrow$ area under the curve

Example:

A rock tossed off a building 300 ft. tall falls at a velocity of $v(t) = -32t + 64$ ft/s. How far does it fall in the first 3 seconds?



Change in position = area under curve

$$= \int_0^3 v(t)\,dt = 64 - 16 = 48 \text{ ft.}$$

After 3 seconds, the rock is 48 ft higher than when it started: $\Delta s = 48$ ft.

How else can we compute this? If $s(t)$ is the position at time t, then $s'(t) = v(t)$ represents the velocity. What we really want to compute is $\Delta s = s(3) - s(0)$.

Since s is an antiderivative of v,

$$\int_0^3 v(t)\,dt = \int_0^3 (-32t + 64)\,dt = \left(\frac{-32t^2}{2} + 64t\right)\Bigg|_0^3$$

$$= (-16 \cdot 9 + 64 \cdot 3) - (0 + 0) = -144 + 192 = 48 \text{ ft}$$

The function $-16t^2 + 64t$ is one of many antiderivatives of $-32t + 64$.

This "family of antiderivatives" is represented by an indefinite integral:

$$\int v(t)\,dt = \int (-32t + 64)\,dt = -16t^2 + 64t + C$$

This expression represents *all* antiderivatives, but which one is the correct one for this problem?

The initial height of the rock is $s(0) = 300$ ft. We know

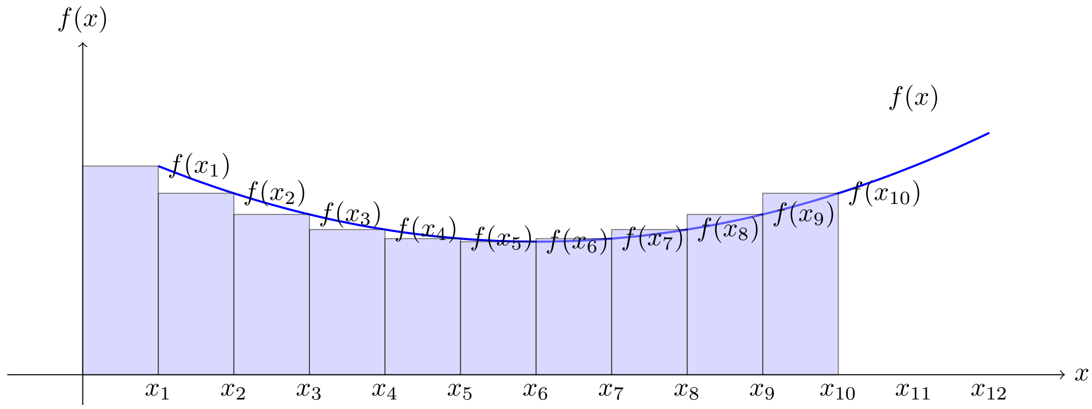
$$300 = s(0) = -16t^2 + 64t + C\Big|_{t=0} = C$$

$\therefore C = 300$

Thus, the position of the rock as a function of time is given by:

$$s(t) = -16t^2 + 64t + 300$$

The way we actually define the integral/area under a curve is via a Riemann sum:



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \text{"approximation by } n \text{ rectangles"}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} A_k \quad \text{where } A_k = (\Delta x)\cdot f(x_k)$$

$$\left(\Delta x = \frac{b-a}{n}, \quad x_k = a + k\,\Delta x \ldots\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \Delta x$$

Δx: "change in x"
dx: "infinitesimal change in x"

The Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{(this is a definite integral: a number)}$$

$$\int f(x)\,dx = F(x) + C \quad \text{(this is an indefinite integral: family of functions)}$$

Example:

$$\int \cos(x)\,dx = \sin(x) + C$$

$$\int_0^{\pi/4} \cos(x)\,dx = \sin(x)\Big|_{x=0}^{x=\pi/4}$$

$$= \sin\left(\frac{\pi}{4}\right) - \sin(0) = \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

Rules for Integration:

- **The Sum Rule:** $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- **The Constant Multiple Rule:** $\int c \cdot f(x)\,dx = c\int f(x)\,dx$
- **Power Rule:** $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

- **Domain Rules:** $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

- **Caution:** $\int f(x)g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$

Modeling Problems:

Sometimes we need the formula:

$$F(t) = F(a) + \int_a^t f(x)\,dx$$

where $f(t)$ is the rate of change of $F(t)$ (i.e., $F'(t) = f(t)$).

Example: A culture of cells in a lab has a population of 100 when nutrients are added at time $t = 0$. Suppose the population $N(t)$ increases at a rate of

$$90e^{-0.1t} \text{ cells/hour} = N'(t)$$

Find a formula for $N(t)$. What happens to the population "in the long run"?

$$N(t) = N(0) + \int_0^t N'(x)\,dx \quad \text{(where } x \text{ is a "dummy variable")}$$

$$= 100 + \int_0^t 90e^{-0.1x}\,dx$$

Note: The antiderivative of e^{ax} is $\frac{1}{a}e^{ax}$ for any number a

$$= 100 + 90 \cdot \frac{1}{-0.1} e^{-0.1x}\Big|_{x=0}^{x=t}$$

$$= 100 - \left[900e^{-0.1x}\right]\Big|_{x=0}^{x=t}$$

$$= 100 - \left(900e^{-0.1t} - 900e^{0}\right)$$

$$= 100 - 900e^{-0.1t} + 900$$
$$= \boxed{1000 - 900e^{-0.1t}}$$

What happens "in the long run"?

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \left(1000 - 900e^{-0.1t}\right) = 1000 \text{ cells}$$